

# A quantum algorithm for the quantum Schur-Weyl transform

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# A quantum algorithm for the quantum Schur-Weyl transform

## **Abstract**

We construct an efficient quantum algorithm to compute the quantum Schur-Weyl transform for any value of the quantum parameter  $q \in [0, \infty]$ . Our algorithm is a  $q$ -deformation of the Bacon-Chuang-Harrow algorithm [1], in the sense that it has the same structure and is identically equal when  $q = 1$ . When  $q = 0$ , our algorithm is the unitary realization of the Robinson-Schensted-Knuth (or RSK) algorithm, while when  $q = \infty$  it is the dual RSK algorithm together with phase signs. Thus, we interpret a well-motivated quantum algorithm as a generalization of a well-known classical algorithm.

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# Chapter 1

## Introduction

This thesis addresses some problems in quantum computation that are motivated by quantum algebra.

The quantum Fourier transform for a finite group  $G$  plays a central role in the theory of quantum algorithms. This is another name for the Burnside decomposition of the group algebra of  $G$ ,

$$\mathbb{C}[G] \cong \bigoplus_V V \otimes V^*,$$

which is an isomorphism of Hilbert spaces as well as an isomorphism of algebras. Since the Burnside decomposition is a Hilbert space isomorphism and therefore a unitary operator, one can ask when it can be expressed by a small quantum circuit, or equivalently, when it has a fast quantum algorithm.

Polynomial-time quantum algorithms for the Burnside decomposition are known for many finite groups (see for example [16],[2],[12]). In especially favorable cases, the quantum Fourier transform for  $G$  yields an algorithm for the hidden subgroup problem for  $G$  or other groups related to  $G$ . In particular, the Shor-Simon-Kitaev algorithm (see [16], [17], [9]) to find periods or compute discrete logarithms in any

finitely generated abelian group is based on the quantum Fourier transform for finite abelian groups.

The Schur-Weyl decomposition is another transform which is related to the Burnside decomposition for the symmetric group  $S(n)$ . Given a Hilbert space or *qudit*  $V = \mathbb{C}^d$ , the Schur-Weyl decomposition is

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} R^\lambda \otimes V^\lambda, \quad (1.0.1)$$

where  $R^\lambda$  is an irreducible representation of the symmetric group  $S(n)$ , acting by permuting tensor factors, while  $V^\lambda$  is an irreducible representation of the unitary group  $U(d)$ , which acts simultaneously or diagonally on all of the factors of  $V$ . The fact that  $R^\lambda$  is the multiplicity space of  $V^\lambda$  and vice versa is known as Schur-Weyl duality. Recently, Bacon, Chuang, and Harrow presented an efficient quantum algorithm to compute a basis refinement of this decomposition [1].

In this thesis, we clarify and generalize the Bacon-Chuang-Harrow (or BCH) algorithm. First, the Schur-Weyl decomposition has a generalization that depends on a parameter  $q$  from quantum algebra. We replace the unitary group  $U(d)$  with the quantum group  $U_q(d) = U_q(\mathfrak{gl}(d))$ , and the symmetric group  $S(n)$  with the Hecke algebra  $H_q(n)$ . Then the Schur-Weyl decomposition still exists for every  $q \in \mathbb{C}$  which is not a root of unity, but the specific linear isomorphism expressed by equation 1.0.1 depends on  $q$ . (If  $q$  is a root unity of order  $r$ , then the decomposition still exists, but it degenerates into a different form when  $r = O(n + k)$ .) When  $q$  is real and positive, then both sides of 1.0.1 are naturally Hilbert spaces and the isomorphism is still unitary.

Our main result is the following theorem which appears in Section 6.4.



**Theorem 1.0.1.** *There is an efficient continuous family of quantum algorithms for the quantum Schur-Weyl transform for each  $q \in [0, \infty]$ . When  $q = 1$ , the algorithm is the Bacon-Chuang-Harrow algorithm. The algorithm continuously extends to  $q = 0$  and becomes a unitary form of the Robinson-Schensted-Knuth (RSK) algorithm ([15], [10]) together with phase signs. The algorithm also continuously extends to  $q = \infty$ , and becomes the dual RSK algorithm without any phases.*

Note the double use of the word “quantum”, referring to both quantum computation and quantum algebra. Those constructions in quantum algebra that are non-unitary have no quantum computation interpretation, while many constructions in quantum computation only have a pro forma interpretation as quantum algebra. Theorem 1.0.1 properly lies in both topics. In fact, the two senses of quantumness are slightly incongruous. In quantum algebra, the  $q = 1$  case is called classical or non-quantum, because it is the case in which quantum groups become ordinary groups. But as an algorithm, the Schur-Weyl transform is not classical when  $q = 1$ ; it becomes classical when  $q = 0$  instead. The limit  $q = 0$  is called the *crystal limit* in quantum algebra.

Like the BCH algorithm, our algorithm has running time polynomial in the number of qudits  $n$ , the size of the qudit  $d$ , and  $\log \epsilon^{-1}$ , where  $\epsilon$  is the desired accuracy. The bound on running time is also uniform in  $q$ , assuming that  $q$  itself can be computed quickly. Therefore, our algorithm is efficient in the sense that it is polynomial in the number of qudits, for any fixed size of qudit. We do not know whether there is an algorithm which is jointly polynomial in  $n$  and  $\log d$ , *i.e.*, polynomial in the input qubit length  $n(\log d)$ .

Our algorithm can be compared to quantum straightening algorithms [11]. Our algorithm can be called a Schur-Weyl straightening algorithm, but we emphasize

a different interpretation. Straightening algorithms are traditionally interpreted as algorithms in symbolic algebra or numerical analysis. As such, the input is not a linear number of qubits or qudits, but rather an exponential list of components of a vector in a vector space such as  $V^{\otimes n}$ . One can make the same distinction between a quantum Fourier transform and a classical discrete Fourier transform, which can be algebraically the same, but are interpreted differently as computer science. One interesting connection between the two interpretations is that a polynomial-time algorithm for a quantum transform always yields a quasilinear-time algorithm for the corresponding numerical transform. (The converse does not hold in general.)

Finally, in our interpretation and proof of 1.0.1, we will be more precise about the basis refinement of 1.0.1. The relevant basis of  $V^\lambda$ , which can be called the “insertion tableau” by extension from the  $q = 0$  and  $q = \infty$  cases, is its Gelfand-Tsetlin-Jimbo (or GTJ) basis. The BCH algorithm and our generalization compute their result in this basis essentially by construction — the algorithm is built from a subroutine, the Pieri transform (which BCH call the Clebsch-Gordan transform), that stays in this basis. We will also prove that the algorithm yields the Young-Yamanouchi-Hoefsmit (or YYH) basis of  $R^\lambda$ , up to sign. (Bacon, Chuang, and Harrow state that it produces the Young-Yamanouchi basis without proof.) Finally, specific bases of each  $V^\lambda$  and  $R^\lambda$  do not quite completely determine a basis of the right side of equation 1.0.1, because we could still multiply each summand  $R^\lambda \otimes V^\lambda$  by a scalar, or in the Hilbert space case, by a phase. In this sense, the Schur-Weyl transform is not quite uniquely determined.

This thesis is structured as follows. In Chapter 2 we describe the representation theory of Hopf algebras. In particular, we focus on the Hopf algebra of interest in this thesis, the quantum group  $U_q(d)$ . In Chapter 2 we also investigate the Gelfand

Tsetlin type bases for representations, which have properties desirable for quantum computation. In Chapter 3 we describe the necessary combinatorics to discuss the representation theory defining the Schur-Weyl transform. We also describe the RSK algorithm and a generalization, which we call quantum insertion. In Chapter 4 we describe the representation theories of the quantum group  $U_q(d)$  and the type A Hecke algebra  $H_q(n)$ , using the combinatorial language detailed in Chapter 3. We end the chapter with the formulation of Schur-Weyl duality, which is central to this thesis. In Chapter 5 we define the Pieri and Schur-Weyl transforms with an emphasis on their connections with insertion algorithms. Finally, in Chapter 6 we give an introductory background to quantum probability and algorithms, and present our main theorem 1.0.1.

## Chapter 2

# Hopf algebra representation theory

### 2.1 Introduction

Finite groups and semisimple Lie algebras are familiar examples of algebraic structures with nice representation theories. Hopf algebras have an algebraic structure which generalizes that of both finite groups and semisimple Lie algebras, while retaining the key properties of their representations. In this chapter we describe Hopf algebras and the basics of their representation theory. In Section 2.2 we define Hopf algebras and the property of cocommutativity. In Section 2.3 we see an example of a noncocommutative Hopf algebra which will reappear in subsequent chapters. In Section 2.4 we define the representation theory of Hopf algebras and state some of the key theorems in their study. Finally, in Section 2.5 we describe a basis for algebra representations which is of both algebraic and computational interest.

## 2.2 Hopf algebras

The material in this section can be found in [8]. An algebra  $A$  over a field  $\mathbb{F}$  has an associative multiplication  $m: A \otimes A \rightarrow A$ . It also has a two-sided unit, which can be expressed as a map  $\iota: \mathbb{F} \rightarrow A$  such that  $\iota(1) \cdot a = a = a \cdot \iota(1)$  for all  $a$  in  $A$ . In pictures, the following two diagrams should commute:

(Associativity Axiom)

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{m \otimes \text{id}} & A \otimes A \\
 \downarrow \text{id} \otimes m & & \downarrow m \\
 A \otimes A & \xrightarrow{m} & A
 \end{array}$$

(Unit Axiom)

$$\begin{array}{ccccc}
 \mathbb{F} \otimes A & \xrightarrow{\iota \otimes \text{id}} & A \otimes A & \xleftarrow{\text{id} \otimes \iota} & A \otimes \mathbb{F} \\
 & \searrow \cong & \downarrow m & \swarrow \cong & \\
 & & A & & 
 \end{array}$$

A *coalgebra* is obtained by reversing all the arrows. Thus we have a coassociative comultiplication  $\Delta: A \rightarrow A \otimes A$  and a two-sided counit  $\varepsilon: A \rightarrow \mathbb{F}$ , where the following two diagrams should commute:

(Coassociativity Axiom)

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xleftarrow{\Delta \otimes \text{id}} & A \otimes A \\
 \uparrow \text{id} \otimes \Delta & & \uparrow \Delta \\
 A \otimes A & \xleftarrow{\Delta} & A
 \end{array}$$

(Counit Axiom)

$$\begin{array}{ccccc}
 \mathbb{F} \otimes A & \xleftarrow{\varepsilon \otimes \text{id}} & A \otimes A & \xrightarrow{\text{id} \otimes \varepsilon} & A \otimes \mathbb{F} \\
 & \swarrow \cong & \uparrow \Delta & \searrow \cong & \\
 & & A & & 
 \end{array}$$

If an algebra  $A$  also has a coalgebra structure, so that the maps  $\Delta, \varepsilon$  are algebra homomorphisms and  $m, \iota$  are coalgebra homomorphisms, then it is a *bialgebra*. Then, a *Hopf algebra* is a bialgebra with a map called the *antipode*. The antipode is a bialgebra endomorphism  $S: A \rightarrow A$  where the following three compositions are identical:

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{S \otimes \text{id}} A \otimes A \xrightarrow{m} A$$

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\text{id} \otimes S} A \otimes A \xrightarrow{m} A$$

$$A \xrightarrow{\varepsilon} \mathbb{F} \xrightarrow{\iota} A$$

We will generally consider algebras over the complex numbers  $\mathbb{C}$  and the real numbers  $\mathbb{R}$ .

**Example 2.2.1.** Given a group  $G$ , we can form its group algebra  $\mathbb{C}[G]$  with basis indexed by elements  $g \in G$ .  $\mathbb{C}[G]$  is in fact a Hopf algebra with coproduct, counit, and antipode map defined by

$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1$$

$$S(g) = g^{-1}$$

**Example 2.2.2.** Let  $\mathfrak{g}$  be a Lie algebra over  $\mathbb{C}$ . Then its universal enveloping algebra

$U(\mathfrak{g})$  is a Hopf algebra with coproduct, counit, and antipode map defined by

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

$$\varepsilon(a) = 0$$

$$S(a) = -a$$

If  $A$  is an algebra over  $\mathbb{C}$ , then it is a  $*$ -algebra if it has a map  $*$ :  $A \rightarrow A$  with the following properties:

$$\begin{aligned} (a+b)^* &= a^* + b^* & (\lambda a)^* &= \overline{\lambda} a^* \\ (ab)^* &= b^* a^* & a^{**} &= a, \end{aligned}$$

for  $a, b \in A$  and  $\lambda \in \mathbb{C}$ . If  $A$  is a  $*$ -algebra and Hopf algebra so that  $\Delta(x^*) = \Delta(x)^*$ , then we call  $A$  a *Hopf  $*$ -algebra*.

If  $A_{\mathbb{R}}$  is an algebra over  $\mathbb{R}$ , then  $A_{\mathbb{C}} = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$  is an algebra over  $\mathbb{C}$ . On the other hand a complex algebra  $A_{\mathbb{C}}$  may have more than one decomplexification  $A_{\mathbb{R}}$ , even though there is always an obvious algebra inclusion  $A_{\mathbb{R}} \subseteq A$ .

Specifying a decomplexification of  $A$  is equivalent to choosing a “bar structure”  $a \mapsto \overline{a}$  that satisfies the axioms:

$$\begin{aligned} \overline{a+b} &= \overline{a} + \overline{b} & \overline{\lambda a} &= \overline{\lambda} \overline{a} \\ \overline{ab} &= \overline{a} \overline{b} & \overline{\overline{a}} &= a. \end{aligned}$$

This is almost the same as a  $*$ -structure, the difference being that a bar structure does not reverse multiplication. Given a bar structure on  $A$ , the real subalgebra  $A_{\mathbb{R}}$  is the set of self-conjugate elements  $a = \overline{a}$ . Also, if  $A$  has both a  $*$ -structure and a

bar structure, then we require that they commute, or

$$(\overline{a})^* = \overline{(a^*)}.$$

In some cases, such as for  $\mathbb{C}[G]$ , the antipode map is involutory, and the  $*$ -map is essentially a conjugate-linear version of the antipode map. In other cases, the antipode map will not be involutory and there is some other  $*$ -map making the algebra into a Hopf  $*$ -algebra.

We end by describing the condition of commutativity and define the analogue for the coalgebraic structure of a Hopf algebra. Let  $A$  be an algebra with multiplication map  $m$ . Define the flip map  $\tau: A \otimes A \rightarrow A \otimes A$  by  $\tau(x \otimes y) = y \otimes x$ . One way of defining  $A$  to be commutative is by requiring the following diagram commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & \searrow m \quad \swarrow m & \\ & A & \end{array}$$

Thus, if  $A$  is a coalgebra with comultiplication map  $\Delta$ , we define *cocommutativity* by requiring the following diagram commute instead:

$$\begin{array}{ccc} A \otimes A & \xleftarrow{\tau} & A \otimes A \\ & \swarrow \Delta \quad \searrow \Delta & \\ & A & \end{array}$$

Note that both  $\mathbb{C}[G]$  and  $U(\mathfrak{g})$  are generally noncommutative but always cocommutative. In the next section we'll examine a Hopf algebra which is both noncommutative and noncocommutative.



## 2.3 The quantum group of $\mathfrak{gl}(d)$

The Hopf algebras  $\mathbb{C}[G]$  and  $U(\mathfrak{g})$  we saw in section 2.2 are generally noncommutative but always cocommutative. In this section, we introduce an example of a Hopf algebra which is noncommutative and noncocommutative: the *quantum group*. Quantum groups as defined independently by Drinfeld [5] and Jimbo [7] are deformations of  $U(\mathfrak{g})$  for  $\mathfrak{g}$  a Lie algebra.

In this section we consider the Lie algebra  $\mathfrak{gl}(d)$  which is isomorphic to  $\text{End}(d)$ , the set of linear maps on  $\mathbb{C}^d$ . The generators of  $U(\mathfrak{gl}(d))$  are  $e_i$  and  $f_i$  for  $1 \leq i \leq d-1$ , and  $h_i$  for  $1 \leq i \leq d$ . The relations on the generators are called Serre relations and are given by:

$$\begin{aligned} [h_i, h_j] &= 0 && \text{for } j \neq i \\ [h_i, f_j] &= [h_i, e_j] = 0 \\ [e_i, f_j] &= \delta_{ij} h_i \\ [e_i, e_j] &= [f_i, f_j] = 0 && \text{for } |i - j| > 1 \\ e_i e_{i\pm 1} e_i &= \frac{1}{2} (e_i^2 e_{i\pm 1} + e_{i\pm 1} e_i^2) \\ f_i f_{i\pm 1} f_i &= \frac{1}{2} (f_i^2 f_{i\pm 1} + f_{i\pm 1} f_i^2). \end{aligned}$$

The associated Drinfeld-Jimbo quantum deformation of  $U(\mathfrak{gl}(d))$  is called a *quantum group*, and is written  $U_q(\mathfrak{gl}(d))$  which we will abbreviate to  $U_q(d)$ . The parameter  $q$  is a complex number not equal to zero or one. The generators of  $U_q(d)$  are  $e_i$  and  $f_i$  for  $1 \leq i \leq d-1$ , and  $q^{\pm h_i/2}$  for  $1 \leq i \leq d$ . The generators  $q^{\pm h_i/2}$  can be interpreted as formal exponentials rather than actual powers of  $q$ . The formal notation is meant to imply that these generators commute with each other and that  $q^{-h_i/2}$  is

the reciprocal of  $q^{h_i/2}$ , using addition in the exponent. In these formal exponentials, we also let  $k_i = h_i - h_{i+1}$ .

We use the notation  $[n]$  for the *quantum integer* defined by the formula

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-(n-3)} + q^{-(n-1)}.$$

Extending the notation to operators, we write  $[h_i] = \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$ .

The relations on the  $U_q(d)$  generators are  $q$ -deformations of the  $U(\mathfrak{g})$  Serre relations, and are given by

$$\begin{aligned} [q^{h_i/2}, q^{h_j/2}] &= 0, \text{ for } i \neq j \\ q^{h_i/2} e_j &= \begin{cases} q^{1/2} e_j q^{h_i/2} & \text{for } i = j \\ q^{-1/2} e_j q^{h_i/2} & \text{for } i = j + 1 \\ e_j q^{h_i/2} & \text{otherwise} \end{cases} \\ q^{h_i/2} f_j &= \begin{cases} q^{-1/2} f_j q^{h_i/2} & \text{for } i = j \\ q^{1/2} f_j q^{h_i/2} & \text{for } i = j + 1 \\ f_j q^{h_i/2} & \text{otherwise} \end{cases} \\ [e_i, f_j] &= \delta_{ij} [h_i] \\ [e_i, e_j] &= [f_i, f_j] = 0, \quad |i - j| \geq 2 \\ e_i e_{i\pm 1} e_i &= \frac{1}{[2]} (e_i^2 e_{i\pm 1} + e_{i\pm 1} e_i^2) \\ f_i f_{i\pm 1} f_i &= \frac{1}{[2]} (f_i^2 f_{i\pm 1} + f_{i\pm 1} f_i^2). \end{aligned}$$

Interestingly, the Hopf algebra structure on  $U(\mathfrak{g})$  can also be deformed so that

$U_q(d)$  is a Hopf algebra. For example, the coproduct map becomes

$$\begin{aligned}\Delta(q^{h_i/2}) &= q^{h_i/2} \otimes q^{h_i/2} \\ \Delta(e_i) &= e_i \otimes q^{-k_i/2} + q^{k_i/2} \otimes e_i \\ \Delta(f_i) &= f_i \otimes q^{-k_i/2} + q^{k_i/2} \otimes f_i.\end{aligned}$$

There are other deformations of the Hopf algebra structure that result in different coproduct maps. For example, we could replace  $\Delta$  as defined above by  $\tau \circ \Delta$ , which is distinct from  $\Delta$  by noncocommutativity.

When  $q$  is real and positive,  $U_q(d)$  has a  $*$ -map defined by

$$e_i^* = f_i \quad f_i^* = e_i \quad (q^{h_i/2})^* = q^{h_i/2}.$$

This  $*$ -map makes  $U_q(d)$  into a Hopf  $*$ -algebra. When  $q$  is real and positive,  $U_q(d)$  also has a bar structure in which all of the generators are real, and they generate a real Hopf  $*$ -algebra  $U_q(d)_{\mathbb{R}}$ .

In future sections we'll restrict to the case when  $q$  is real and positive so that we can use the associated  $*$  and bar structures.

## 2.4 The representation theory of Hopf algebras

A *representation* of an algebra  $A$  is a vector space  $V$  and a linear map  $\rho : A \rightarrow \text{End}(V)$  which preserves the multiplication and unity, i.e.  $\rho(ab) = \rho(a)\rho(b)$  and  $\rho(1) = 1$ . The action  $\rho$  can be implied so that  $\rho(a)v$  is written  $av$ , or in quantum notation as  $a|v\rangle$ . In the rest of this section, we fix the assumptions that our algebra  $A$  is a Hopf algebra, and our representations  $V$  are defined over  $\mathbb{C}$  and are finite dimensional.

Two representations  $V$  and  $W$  of  $A$  are *isomorphic* if there exists a linear bijection  $T : V \rightarrow W$  that commutes with the action of  $A$ , *i.e.*,  $T(av) = aT(v)$  for all  $a \in A$ ,  $v \in V$ .

The representation  $V$  is *irreducible* if it has no non-trivial subspaces that are closed under the action of  $A$ . In this thesis we use the abbreviation *irrep*. For example, the counit map  $\varepsilon : A \rightarrow \mathbb{C}$  defines a *trivial representation*, which is irreducible since it's one dimensional.

Given two representations  $V$  and  $W$  of  $A$ , there is a well-defined representation structure on the direct sum  $V \oplus W$  given by

$$a(v \oplus w) = av \oplus aw.$$

A representation  $V$  is *semisimple* if it is isomorphic to a direct sum of irreps. (Likewise, an algebra  $A$  is called semisimple if all of its representations are semisimple.) The number of occurrences of an irrep  $W$  in  $V$  is called the *multiplicity* of  $W$ . If the multiplicities are all 0 or 1, then  $V$  is called *multiplicity-free*.

Assume for the moment that  $A$  and its subalgebras are semisimple. Given a representation  $V$  of  $A$  and a subalgebra  $B \subseteq A$ , the restriction of  $V$  to  $B$  will be denoted by  $\text{Res}_B^A V$ . When  $V$  is an irrep of  $A$ ,  $\text{Res}_B^A V$  is typically not an irrep of  $B$ , but by semisimplicity  $\text{Res}_B^A V$  decomposes as a direct sum of irreps of  $B$ . A rule for describing the decomposition of  $\text{Res}_B^A V$  into irreps is called a *branching rule*. If for all irreps  $V$  of  $A$ , the branching rule for  $\text{Res}_B^A V$  is multiplicity-free, then the inclusion  $B \subseteq A$  is called a *Gelfand pair*.

Given two representations  $V$  and  $W$  of  $A$ , the coproduct map  $\Delta : A \rightarrow A \otimes A$  is used to define a representation structure on  $V \otimes W$ .

The antipode map is used to define a *dual representation*. Given a representation

$V$  of  $A$ , define  $V^*$  to be the dual space of linear functionals on  $V$ . Then, the action of  $A$  on  $V^*$  is defined by  $a\langle v| = \langle v|S(a)$ .

If  $A$  is a Hopf  $*$ -algebra, then  $V$  is a  $*$ -representation if  $\rho(a^*) = \rho(a)^*$  where the  $*$  on the right side is the Hermitian adjoint. (The Hermitian adjoint makes the algebra  $\text{End}(V)$  into a  $*$ -algebra.) This generalizes the notion of a unitary representation of a group. In particular, a  $*$ -representation  $V$  is automatically semisimple: If  $W$  is a subrepresentation of  $V$ , then so is its orthogonal complement  $W^\perp$ .

Although such a  $V$  might possibly have non-orthogonal irreducible decompositions, it always has an orthogonal irreducible decomposition. If  $V$  is multiplicity-free, then its irreducible decomposition is unique and therefore orthogonal.

Our analysis so far carries over verbatim to representations of algebras over  $\mathbb{R}$ . Quantum computation is defined over  $\mathbb{C}$ , and we will ultimately be interested in connecting representations over  $\mathbb{R}$  with representations over  $\mathbb{C}$ .

If  $V_{\mathbb{R}}$  is a representation of  $A_{\mathbb{R}}$ , then

$$V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

is a bar representation of  $A_{\mathbb{C}}$ . But note that even if  $V_{\mathbb{R}}$  is irreducible,  $V_{\mathbb{C}}$  may or may not be irreducible. If  $\text{End}(V_{\mathbb{R}}) \cong \mathbb{R}$ , then  $V_{\mathbb{C}}$  is irreducible, while if  $\text{End}(V_{\mathbb{R}}) \cong \mathbb{C}$  or  $\text{End}(V_{\mathbb{R}}) \cong H$  (the quaternions), then  $V_{\mathbb{C}}$  has two irreducible summands. In the former case, we will say that  $V_{\mathbb{R}}$  is *strongly irreducible*.

**Example 2.4.1.**  $\mathbb{C}[G]$  is a Hopf  $*$ -algebra with  $*$ -map defined by  $g^* = g^{-1}$ . Note that in this case a representation being a  $*$ -representation is the same thing as it being unitary as a representation of  $G$ . Also  $\mathbb{C}[G]$  has a standard bar structure with  $g = \overline{g}$ , so that its decomplexification is the real group algebra  $\mathbb{R}[G]$ .

When  $\mathbb{C}[G]$  is finite dimensional it has additional properties for its irreps. For example, there are finitely many distinct irreps of  $\mathbb{C}[G]$ , indexed by the conjugacy classes of  $G$ . And we always have semisimplicity of representations of  $\mathbb{C}[G]$ .

**Example 2.4.2.** Every continuous representation  $V$  of a connected Lie group  $G$  is also a representation of the universal enveloping algebra  $U(\mathfrak{g})$  and it has the same subrepresentations.

If  $\mathfrak{g}_{\mathbb{R}}$  is a real Lie algebra and  $\mathfrak{g}_{\mathbb{C}}$  is its complexification, then  $U(\mathfrak{g}_{\mathbb{C}})$  has both a natural bar structure — where the real subalgebra is  $U(\mathfrak{g}_{\mathbb{R}})$  — and a natural  $*$ -structure. Since  $U(\mathfrak{g}_{\mathbb{C}})$  is generated as a complex algebra by  $\mathfrak{g}_{\mathbb{R}}$ , we define these structures by letting

$$\bar{a} = a \quad a^* = -a$$

for  $a \in \mathfrak{g}_{\mathbb{R}}$ .

## 2.5 Gelfand-Tsetlin type bases

In this section, we describe bases for irreps with special algebraic and computational properties. We will be interested in a tower of algebras

$$\mathbb{C} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$$

and we will use the abbreviation

$$\text{Res}_{k-1}^k V = \text{Res}_{A_{k-1}}^{A_k} V$$

for the restriction of a representation  $V$  of  $A_k$ .

Suppose that each inclusion  $A_{k-1} \subseteq A_n$  is a Gelfand pair. Then if  $V = V_n$  is an

irrep of  $A_n$ ,  $\text{Res}_{n-1}^n V$  is a direct sum of irreps  $V_{n-1}$  of  $A_{n-1}$ , and by induction each  $\text{Res}_{k-1}^k V_k$  is a direct sum of irreps  $V_{k-1}$  of  $A_{k-1}$ . As a result,  $V$  is expressed as a direct sum of irreps  $V_0$  of  $A_0 = \mathbb{C}$ , and all such irreps are isomorphic and 1-dimensional. Thus  $V$  has a basis of lines which are encoded by flags

$$\mathbb{C} \cong V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V.$$

This line basis is called a Gelfand Tsetlin type (GTT) basis. By extension, any vector basis that refines the GTT line basis is also called a GTT basis. Note that in the encoding, the number of bits a GTT basis vector requires is the sum of the bits required to encode each summand  $V_k$ .

To get a sense of the significance of a GTT basis, note that whenever  $a \in A_k$  and  $v \in V_k$ , then  $av \in V_k$  also. This means that we can express the action of an element  $a$  on  $V$  in the setting of a lower-dimensional algebra, which naturally gives rise to a recursive structure. However, note that a GTT vector basis of an irrep  $V$  is not unique; only the corresponding line basis is unique. The computational strength of a GTT basis can still depend on how its vectors are scaled.

*Remark 2.5.1.* In some articles in quantum computation, if  $V$  is an irrep of a group  $G$  and  $H \subseteq G$  is a subgroup, then a basis that refines a decomposition of  $\text{Res}_H^G V$  is called *subgroup-adapted*. The analogous notion for us is a basis that is *subalgebra-adapted*. In this terminology, a GTT basis is recursively adapted to a tower of subgroups or subalgebras.

If each algebra  $A_k$  is a  $*$ -algebra and  $V$  is a  $*$ -representation of  $A = A_n$ , then a GTT basis is automatically orthogonal, because each restriction  $\text{Res}_{k-1}^k V_k$  has an orthogonal decomposition. We further require that a GTT vector basis of a  $*$ -representation

be orthonormal, so that the basis is usable in quantum computation. However, even when GTT basis vectors are orthonormal, their phases are still not determined by the GTT property.

If  $V$  and  $W$  are two irreps of an algebra  $A$ , with given GTT bases, then technically their combinations  $V \oplus W$  and  $V \otimes W$  do not have GTT bases. However, we can define standard bases by taking the direct sum and tensor bases, respectively. These combinations are GTT bases with respect to the action of  $A \otimes A$  instead.



## Chapter 3

# The combinatorics of Young tableaux and insertion algorithms

### 3.1 Introduction

The representation theories of the algebras described in this thesis are indexed by combinatorial objects called Young tableaux. In this chapter we describe the combinatorics of these objects. In Section 3.2 we define Young tableaux and state some of their key properties. In Section 3.3 we describe insertion algorithms for operating on Young tableau, which will connect to some interesting representation theory in subsequent chapters.

### 3.2 The combinatorics of Young tableaux

A *partition*  $\lambda$  is a list of non-negative integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$$

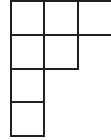
such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d.$$

We say that  $\lambda$  is a partition of  $n$ , or  $\lambda \vdash n$ , if  $\sum_k \lambda_k = n$ . The length of  $\lambda$ , denoted  $\ell(\lambda)$ , is the number of non-zero entries of  $\lambda$ .

A partition  $\lambda$  has an associated *Young diagram*, which is a horizontal histogram with  $\ell(\lambda)$  rows; the  $k$ th row has  $\lambda_k$  boxes.

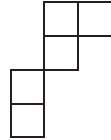
**Example 3.2.1.** The Young diagram of  $\lambda = (3, 2, 1, 1)$  is



If  $\mu$  and  $\lambda$  are partitions so that the Young diagram of  $\mu$  is contained in the Young diagram of  $\lambda$ , then we write  $\mu \subseteq \lambda$ . If  $\lambda$  and  $\mu$  differ by a single box then  $\lambda$  is said to *cover*  $\mu$ .

When  $\mu \subseteq \lambda$  we can form a Young diagram of *skew shape* given by  $\lambda \setminus \mu$  which means removing the boxes in the Young diagram of  $\mu$  from the boxes in the Young diagram of  $\lambda$ .

**Example 3.2.2.** If  $\lambda = (3, 2, 1, 1)$ , and  $\mu = (1, 1)$ , then the skew shape  $\lambda \setminus \mu$  is given by



If  $\lambda \setminus \mu$  has at most one box in each of its columns, then it is called a *horizontal strip*.

A *Young tableau* of shape  $\lambda$  (including skew shapes) is a filling of the boxes of the Young diagram of shape  $\lambda$  with positive integers. If  $t$  is a Young tableau of shape  $\lambda$ ,

we write  $\text{sh}(t) = \lambda$ . We will use special types of tableaux called semi-standard and standard Young tableaux.

A Young tableau of shape  $\lambda$  is *semi-standard* (abbreviated SSYT) if its entries weakly increase from left to right and strictly increase from top to bottom.

**Example 3.2.3.** An example of an SSYT with shape  $(3, 2, 1, 1)$  is given by

$$t = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array}.$$

An example of an SSYT with skew-shape  $(3, 2, 1, 1) \setminus (1, 1)$  is given by

$$u = \begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline & 3 & \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array}.$$

We denote the set of SSYT of shape  $\lambda$  with entries in  $\{1, \dots, d\}$  by  $\text{SSYT}(\lambda, d)$ . When the value of  $d$  is obvious, we suppress it and write  $\text{SSYT}(\lambda)$ .

A Young tableau of shape  $\lambda \vdash n$  is *standard* (abbreviated SYT) if its entries are in  $\{1, \dots, n\}$  and strictly increase both from top to bottom and from left to right.

**Example 3.2.4.** An example of an SYT with shape  $(3, 2, 1, 1)$  is given by

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline 6 & & \\ \hline 7 & & \\ \hline \end{array}.$$

An example of an SYT with skew-shape  $(3, 2, 1, 1) \setminus (1, 1)$  is given by

$$u = \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline & 4 & \\ \hline 3 & & \\ \hline 5 & & \\ \hline \end{array}.$$

We denote the set of standard Young tableaux of shape  $\lambda$  by  $\text{SYT}(\lambda)$ .

If  $\nu$  is a horizontal strip, then the SYT  $u$  obtained by filling the Young diagram of  $\nu$  with letters from left-to-right is called *ordered*. (This is not in general defined for skew-tableau, but exists for horizontal strips.)

**Example 3.2.5.** The ordered SYT of the horizontal strip  $\nu = (3, 1) \setminus (1)$  is given by

$$u = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline 1 & & \\ \hline \end{array}.$$

Given any  $t \in \text{SSYT}(\lambda, d)$ , let  $t^{(k)}$  be the restricted tableau in  $\text{SSYT}(\lambda, k)$  obtained by removing all boxes from  $t$  with numbers larger than  $k$ .

**Example 3.2.6.** Let

$$t = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 & & \\ \hline \end{array}.$$

Then

$$t^{(2)} = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline & & \\ \hline \end{array} \quad t^{(1)} = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & \\ \hline \end{array}.$$

Note that the skew shapes  $sh(t^{(k)}) \setminus sh(t^{(k-1)})$  for an SSYT  $t$  are always horizontal strips for each  $k$ , so we give them the label  $\lambda^{(i)}(t)$  where  $sh(t) = \lambda$ .

Finally, the *residue* of a box  $b$  in a Young tableau  $t$  is the difference of its coordinates. In other words, if  $b$  has coordinates  $(i, j)$ , then its residue is  $res(b) = j - i$ . Two boxes in a Young tableau have the same residue if and only if they lie on the same diagonal.

The *axial distance*  $a$  between two boxes  $b$  and  $b'$  is defined to be the difference in their residues, given by  $a = res(b) - res(b')$ . It's described as a distance because it counts the number of boxes in any path in the Young diagram from the box  $b$  to the box  $b'$  where moves left and down count for  $+1$  and moves right and up count for  $-1$ .

**Example 3.2.7.** Consider the SYT

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & & \\ \hline \end{array}.$$

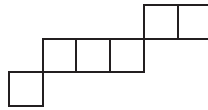
The axial distance from the box containing three to the box containing four is 3. Note the axial distance is antisymmetric, so the distance from the box containing four to the box containing three is  $-3$ .

Given a horizontal strip  $\lambda$ , we define  $a_{ij}$  for  $i < j$  to be the axial distance from the last box in the  $i$ th row to the last box in the  $j$ th row. (Note we can equivalently use the boxes in the second-to-last, third-to-last, etc., positions, and get the same values.)

We will see the axial distances  $a_{ij}$  in subsequent chapters when we describe matrix coefficients that derive from representation theory. One main reason these axial distances are chosen is that they sum in a very natural way:

$$a_{ij} + a_{jk} = a_{ik}.$$

**Example 3.2.8.** If  $\nu$  is the shape given by



then  $a_{12} = 3$ ,  $a_{23} = 4$ , and  $a_{13} = a_{12} + a_{23} = 7$ .

### 3.3 Insertion algorithms

Given an SSYT  $t$  and a letter  $i$ , we will add a new box to  $sh(t)$  to make space for an extra letter and insert the  $i$  into the tableau  $t$ , possibly rearranging other letters in the process. In this section we review the well-known RSK insertion algorithm and introduce a generalization that we call quantum insertion. The RSK algorithm can be found in [10] and [15], while quantum insertion is our way of describing the techniques found in [3].

#### 3.3.1 RSK insertion

The first insertion algorithm we examine is called *Robinson-Schensted-Knuth* (abbreviated RSK), and denoted  $(i \xrightarrow{\text{RSK}} t)$ . This insertion algorithm produces a unique tableau, given by the rules:

1. If  $i$  is greater than or equal to all the numbers in the first row of  $t$ , then add  $i$  to the end of the first row of  $t$ .
2. Otherwise, pick the leftmost box in the first row containing a number  $j > i$ . Replace  $j$  by  $i$ . (This process is referred to as  $i$  *bumping*  $j$ .)
3. Repeat steps (1) and (2) for  $j$  starting with the second row. Proceed inductively.

**Example 3.3.1.** Start with

$$t = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array}$$

If we choose to insert a letter  $i \geq 2$ , such as  $i = 4$ , then it will be added to the end

of the first row:

$$(4 \xrightarrow{\text{RSK}} t) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & & \\ \hline 3 & & & \\ \hline 4 & & & \\ \hline \end{array}$$

However, if we choose to insert the letter  $i = 1$ , it will bump the two out of the first row:

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 3 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array}$$

The two will then bump the three out of the second row, which will itself get added at the end of the third row. Therefore,

$$(1 \xrightarrow{\text{RSK}} t) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 4 & & \\ \hline \end{array}$$

There is also a dual RSK algorithm, denoted  $(i \xrightarrow{\text{RSK}^*} t)$ , which can be thought of as the standard RSK algorithm applied to columns instead of rows. Thus, the dual RSK algorithm also produces a unique output, given by the rules:

1. If  $i$  is larger than all numbers in the first column of  $t$ , add  $i$  to the end of the first column of  $t$ .
2. Otherwise, pick the topmost box in the first column that contains a number  $j \geq i$ . Replace  $j$  by  $i$ .
3. Repeat steps (1) and (2) for  $j$  starting with the second column. Proceed inductively.

Given a word  $w = w_1 \dots w_n$ , we can extend the algorithm by induction to define

$$w \xrightarrow{\text{RSK}} = w_n \xrightarrow{\text{RSK}} (w_{n-1} \xrightarrow{\text{RSK}} (\dots (w_2 \xrightarrow{\text{RSK}} w_1) \dots))$$

**Example 3.3.2.** If  $w = \textcolor{red}{2}1\textcolor{blue}{2}$ ,

$$w \xrightarrow{\text{RSK}} = \begin{array}{|c|c|} \hline 1 & \textcolor{blue}{2} \\ \hline \textcolor{red}{2} & \\ \hline \end{array}$$

Given the word  $w' = \textcolor{red}{2}\textcolor{blue}{2}1$  we still obtain the same output as  $w$ , i.e.  $w \xrightarrow{\text{RSK}} = w' \xrightarrow{\text{RSK}}$ . On the other hand,  $w'' = 1\textcolor{red}{2}\textcolor{blue}{2}$  results in a different tableau

$$w'' \xrightarrow{\text{RSK}} = \begin{array}{|c|c|c|} \hline 1 & \textcolor{red}{2} & \textcolor{blue}{2} \\ \hline \end{array}$$

The above example shows that the RSK map is not invertible. However, note there is a way of distinguishing  $w \xrightarrow{\text{RSK}}$  and  $w' \xrightarrow{\text{RSK}}$  by a *recording tableau* which tracks the order new boxes are added in the sequence of insertions, as is done in the following example. Then we define  $\text{RSK}(w) = P(w) \times Q(w)$ , where  $P(w) = w \xrightarrow{\text{RSK}}$  and  $Q(w)$  is the recording tableau.

**Example 3.3.3.** If  $w = \textcolor{red}{2}1\textcolor{blue}{2}$ ,

$$\text{RSK}(w) = \begin{array}{|c|c|} \hline 1 & \textcolor{blue}{2} \\ \hline \textcolor{red}{2} & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$$

whereas if  $w' = \textcolor{red}{2}\textcolor{blue}{2}1$ ,

$$\text{RSK}(w') = \begin{array}{|c|c|} \hline 1 & \textcolor{blue}{2} \\ \hline \textcolor{red}{2} & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$$

so that  $\text{RSK}(w) \neq \text{RSK}(w')$ .

The proof of the following theorem that the RSK map is a bijection can be found in [10].



**Theorem 3.3.4.** *Let  $V_d^n$  be the set of words in  $d$  letters of length  $n$ . Then, the RSK map  $_{RSK}(w) = P(w) \times Q(w)$  is a bijection between  $V_d^n$  and the disjoint union*

$$\coprod_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} SSYT(\lambda) \times SYT(\lambda).$$

### 3.3.2 Quantum insertion

In this subsection we consider a generalization of RSK we call *quantum insertion*, or q-insertion, and denoted  $(i \xrightarrow{q\text{INS}} t)$ . Given an SSYT  $t$  and a letter  $i$ , q-insertion produces a set of output tableaux, one of which is  $(i \xrightarrow{RSK} t)$ . The rules for constructing the output tableaux are given by:

1. In all possible ways add a new box to  $sh(t)$ .
2. In all possible ways, take either of the following two steps.
  - Insert  $i$  into the new box. If this step is taken, the algorithm terminates.
  - For any letter  $j > i$ ,  $i$  can replace (or bump)  $j$ . In this case step 2 is repeated inductively with  $j$ .

**Example 3.3.5.** Let

$$t = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 4 & & \\ \hline \end{array},$$

and suppose we wish to insert a 2 after adding a box to the second row.

Then, starting with

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 4 & & \\ \hline \end{array},$$

we insert a **2**. It replaces the **3** because it can't take over the new box, and it can't replace the **4**.

1	1	2
2	<b>2</b>	
<b>4</b>		

We then repeat the procedure with the **3**, which can either take over the new box or replace the **4**.

$$\left( 2 \xrightarrow{\text{qINS}} \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 4 & & \\ \hline \end{array} \right) = \left\{ \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & \\ \hline 2 & \mathbf{2} & \mathbf{3} & \\ \hline 4 & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & \\ \hline 2 & \mathbf{2} & \mathbf{4} & \\ \hline \mathbf{3} & & & \\ \hline \end{array} \right\}$$

We define a *bumping sign* for an output tableau as follows. For each letter involved in the bumping procedure, multiply the bumping sign by a  $-1$  if the letter moves to a lower row in the tableau. Note that for RSK, the bumping sign can be  $\pm 1$  whereas for dual RSK the bumping sign is always  $+1$ .

Analogous to the RSK map, given a word  $w = w_1 \dots w_n$ , we extend the q-insertion algorithm by induction to define

$$w \xrightarrow{\text{qINS}} = w_n \xrightarrow{\text{qINS}} (w_{n-1} \xrightarrow{\text{qINS}} (\dots (w_2 \xrightarrow{\text{qINS}} w_1) \dots)).$$

Unlike the RSK algorithm, if  $w'$  is a permutation of  $w$ , then the sets  $w \xrightarrow{\text{qINS}}$  and  $w' \xrightarrow{\text{qINS}}$  are equal. Also unlike RSK, the output of  $w \xrightarrow{\text{qINS}}$  is an entire set of SSYT, and sometimes there is more than one insertion path in  $w \xrightarrow{\text{qINS}}$  which produces an SSYT  $t$ .

As we see in the following example, we can distinguish outputs by attaching a recording tableau which tracks the order in which new boxes are added during the insertion process. Then we define  $\text{qINS}(w) = \{P_q(w) \times Q_q(w)\}$ , where  $P_q(w)$  is a SSYT

in  $w \xrightarrow{\text{qINS}}$  and  $Q_q(w)$  is the associated recording tableau.

**Example 3.3.6.** Letting  $w = 1\bar{2}2$ ,

$$w \xrightarrow{\text{qINS}} = \left\{ \begin{array}{|c|c|c|} \hline 1 & \bar{2} & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & \bar{2} \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \bar{2} & \\ \hline \end{array} \right\}$$

So, the second and third tableaux in  $w \xrightarrow{\text{qINS}}$  are equal. However, we can distinguish the output tableaux by attaching a recording tableau to each:

$$\text{qINS}(w) = \left\{ \begin{array}{|c|c|c|} \hline 1 & \bar{2} & 2 \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & \bar{2} \\ \hline 2 & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \bar{2} & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \right\}$$

Thus far, the reason for using the word “quantum” in the context of a combinatorial insertion algorithm is unclear. In the rest of this section we describe the reason for this choice. Much of the material can be found in [3].

Define the *weighted  $q$ -insertion* map by

$$\text{qINS}(w) = \sum c_{P,Q} P_q(w) \times Q_q(w),$$

for a choice of nonzero constants  $c_{P,Q} \in \mathbb{C}[q, q^{-1}]$ .

The choice of coefficients  $c_{P,Q}$  that interest us is determined by representation theory and will be described in chapter 5. The connection between RSK and  $q$ -insertion becomes clear in the following theorem, which can also be found in [3].

**Theorem 3.3.7.** *Let  $V_d$  be the vector space over  $\mathbb{C}[q, q^{-1}]$  with basis  $\{1, \dots, d\}$ , and consider the vector space  $V_d^{\otimes n}$  of words of length  $n$ . Let  $V^\lambda$  and  $R^\lambda$  be the vector spaces over  $\mathbb{C}[q, q^{-1}]$  with bases  $\text{SSYT}(\lambda)$  and  $\text{SYT}(\lambda)$ , respectively.*

*Then, there exists a choice of coefficients  $c_{P,Q}$  so that the weighted  $q$ -insertion*

$map_{\textit{INS}}(w) = \sum c_{P,Q} P_q(w) \times Q_q(w)$  defines a vector space isomorphism

$$V_d^{\otimes n} \cong \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} V^\lambda \otimes R^\lambda.$$

# Chapter 4

## The representation theories of the quantum group $U_q(d)$ and the Hecke algebra $H_q(n)$

### 4.1 Introduction

In this chapter we describe the representation theories of our quantum algebras of interest. In Section 4.2 we present the representation theory of the quantum group seen in Section 2.3. In Section 4.3 we present the representation theory of the Hecke algebra  $H_q(n)$ , which is a  $q$ -deformation of the symmetric group algebra  $\mathbb{C}[S(n)]$ . These Hecke algebras are not themselves Hopf algebras, but for almost all choices of  $q$  their representations are isomorphic to those of  $\mathbb{C}[S(n)]$ , which is a Hopf algebra. Finally in Section 4.4 we describe the correspondence known as Schur-Weyl duality between representations of the quantum group and Hecke algebra.

## 4.2 The representation theory of $U_q(d)$

We defined the quantum group  $U_q(d)$  in Section 2.3 as an interesting example of a noncommutative and noncocommutative Hopf algebra. In this section we describe its representation theory.

Recall that we restrict the values of  $q$  to real and positive in order to make use of the star and bar structures available in this case. For these values of  $q$ , the irreducible representations of  $U_q(d)$  are isomorphic to those of the unitary group  $U(d)$ . (This is true for other values of  $q$  as well, namely those values of  $q$  which are not roots of unity or zero.) The irreps of  $U(d)$ , and hence the irreps of  $U_q(d)$ , are in bijection with partitions  $\lambda$  whose length is bounded by  $d$ , written  $\ell(\lambda) \leq d$ . For the representation indexed by  $\lambda$  we write  $V^\lambda$ .

Restricting to all the generators except  $e_{d-1}$ ,  $f_{d-1}$ , and  $q^{h_d}$ , we realize a copy of  $U_q(d-1)$  inside  $U_q(d)$ . The branching rule associated to this pairing is given in the following theorem.

**Theorem 4.2.1.** *The algebras  $U_q(d)$  and  $U_q(d-1)$  form a Gelfand pair. In particular, if  $V^\lambda$  is an irrep of  $U_q(d)$ , then*

$$\text{Res}_{n-1}^n V^\lambda = \bigoplus_{\substack{\lambda \setminus \mu \text{ horizontal strip} \\ \ell(\mu) \leq d-1}} V^\mu.$$

The branching rule (4.2.1) implies that a GTT basis for the irrep  $V^\lambda$  can be written  $|v_t\rangle$  with  $t \in \text{SSYT}(\lambda)$ . Then  $V^\lambda$  is the span of the elements  $|v_t\rangle$  so that  $\langle v_t | v_s \rangle = \delta_{t,s}$ .

The specific GTT basis we use is called Gelfand-Tsetlin-Jimbo (GTJ). The formulas described in the rest of this section can be found in [3]. The action of the

generator  $q^{h_i}$  on the GTJ basis is the easiest to describe and is given in the following theorem.

**Theorem 4.2.2.** *Let  $|v_t\rangle$  be a GTJ basis element indexed by the SSYT  $t$ . Then,*

$$q^{h_i/2}|v_t\rangle = q^{x_i(t)/2}|v_t\rangle,$$

where  $x_i(t)$  counts the number of  $i$ 's in  $t$ .

The generator  $f_i$  acts on  $|v_t\rangle$  by turning an instance of  $i$  in the tableau  $t$  into an  $i + 1$  (in all possible ways, i.e. in superposition). In other words, letting  $|v_{t_k}\rangle$  be the vector indexed by tableau  $t_k$  where the last  $i$  in row  $k$  is changed into an  $i + 1$  but is otherwise identical to  $t$ , or zero if this is not possible, then

$$f_i|v_t\rangle = \sum_k \langle v_{t_k} | f_i | v_t \rangle |v_{t_k}\rangle$$

for some choice of coefficients  $\langle v_{t_k} | f_i | v_t \rangle$ , which we call the *GTJ statistic*. The action of  $e_i$  on  $|v_t\rangle$  is also defined with GTJ statistics using the relation  $e_i^* = f_i$ . The GTJ coefficients are complicated notationally, but in principle derive from simple combinatorial properties, in particular axial distances, of the SSYT  $t$ . Recall we defined the axial distance  $a_{ij}$  of  $t$ , and associated horizontal strips  $\lambda^{(i)}(t)$  in section 3.2.

**Theorem 4.2.3.** *Let  $|v_t\rangle$  be a GTT basis element of  $V^\lambda$  indexed by the SSYT  $t$ . Then the GTJ statistic is given by*

$$\langle v_{t_k} | f_i | v_t \rangle = \sqrt{[\lambda_k^{(i)}][\lambda_k^{(i+1)} + 1] \prod_{\substack{j=1 \\ j \neq k}}^{i+1} \frac{[a_{jk} - \lambda_k^{(i)}][a_{jk} + \lambda_k^{(i+1)} + 1]}{[a_{jk}][a_{jk} + 1]}} \quad (4.2.1)$$

### 4.3 The representation theory of the Hecke algebra $H_q(n)$

The Hecke algebra  $H_q(n)$  is a certain  $q$ -deformation of the group algebra  $\mathbb{C}[S(n)]$ . (More precisely, we consider a Iwahori-Hecke algebra of type A. There are also other kinds of Hecke algebras.) Note that the Hecke algebra  $H_q(n)$  for  $q \neq 1$  is *not* a Hopf algebra, so the results in this section are proved independently of the theorems for Hopf algebra representation theory.

The Hecke algebra  $H_q(n)$  with complex parameter  $q$  has generators  $\{T_1, \dots, T_{n-1}\}$  with relations

$$\begin{aligned} T_i T_j &= T_j T_i && \text{for } |i - j| > 1 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\ (T_i - q^{-1})(T_i + q) &= 0. \end{aligned}$$

The first two relations are known as the *braid* relations and the third is the *quadratic* relation. (We use the generators used by Jimbo [7]; the generators due to Iwahori are slightly different.) When  $q = 1$ , the third relation simplifies to  $T_i^2 = 1$ , so that in this case  $T_i$  represents the transposition  $s_i = (i, i + 1)$  in the symmetric group. In other words,  $H_1(n) = \mathbb{C}[S(n)]$ .

As with the quantum group  $U_q(d)$ , we restrict to the case when  $q$  is real and positive. In this case,  $H_q(n)$  has both a  $*$ -structure and a bar structure, defined by

$$T_i^* = T_i \quad \overline{T_i} = T_i.$$

For these values of  $q$ , the irreducible representations of  $H_q(n)$  are isomorphic to



those of  $\mathbb{C}[S(n)]$ . (This is true for other values of  $q$  as well, namely those values of  $q$  which are not roots of unity or zero.) The irreps of  $\mathbb{C}[S(n)]$ , and hence the irreps of  $H_q(n)$  are in bijection with the conjugacy classes of  $S(n)$ , so are indexed by partitions of  $n$ . For the representation indexed by  $\lambda \vdash n$  we write  $R^\lambda$ . Importantly, for these values of  $q$ ,  $H_q(n)$  representations remain semisimple.

It is known that the dimension of  $R^\lambda$  equals the number of standard Young tableaux of shape  $\lambda$  (given by, for example, the hook length formula). Therefore, there is a basis of  $R^\lambda$  indexed by  $\text{SYT}(\lambda)$ . We describe below how GTT bases are naturally described by  $\text{SYT}(\lambda)$ .

The Hecke algebra  $H_q(n)$  contains many copies of  $H_q(n-1)$ ; we consider the one obtained by restricting to the generators  $T_1, \dots, T_{n-2}$ . Thus, we can describe the restriction of  $R^\lambda$  to  $H_q(n)$ . The corresponding branching rule is multiplicity-free and has a nice combinatorial description in terms of the covering relation of Young diagrams.

**Theorem 4.3.1.** *The algebras  $H_q(n)$  and  $H_q(n-1)$  form a Gelfand pair. In particular, if  $R^\lambda$  is an irrep of  $H_q(n)$ , then*

$$\text{Res}_{n-1}^n R^\lambda = \bigoplus_{\lambda \text{ covers } \mu} R^\mu. \quad (4.3.1)$$

Theorem 4.3.1 implies a GTT line basis with elements indexed by sequences of partitions pairwise differing by a single box, i.e. standard Young tableaux. The vector basis of  $R^\lambda$  we use in this thesis that is a refinement of the GTT line basis defined by Theorem 4.3.1 we call the Young-Yamanouchi-Hoefsmit (YYH) basis. We write YYH basis elements as  $|r_t\rangle$  where  $t \in \text{SYT}(\lambda)$ . Then  $R^\lambda$  is the span of the elements  $|r_t\rangle$  with  $\langle r_t | r_s \rangle = \delta_{t,s}$ .

Define the following action of  $S(n)$  on the basis element  $r_t$ :

- If  $i$  and  $i + 1$  are in the same row or column of  $t$  then  $r_{s_i \cdot t} = 0$ .
- Otherwise,  $r_{s_i \cdot t} = r_{t'}$  where  $t'$  is the standard tableau obtained by switching  $i$  and  $i + 1$  in  $t$ .

The action of  $H_q(n)$  on the YYH basis defined below in Formula 4.3.2 is a normalized version of that given in [14], building on that found in [6].

**Theorem 4.3.2.** *Let  $a$  be the axial distance in  $t$  from the box containing  $i$  to the box containing  $i + 1$ . The action of  $H_q(n)$  on  $R^\lambda$  with the YYH basis is defined by*

$$T_i |r_t\rangle = \frac{q^{-a}}{[a]} |r_t\rangle + \sqrt{1 - \frac{1}{[a]^2}} |r_{s_i \cdot t}\rangle \quad (4.3.2)$$

**Example 4.3.3.** Consider  $R^{(2,1)}$  with basis  $\begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}$ . Then,

$$T_1 = \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix}, \quad T_2 = \begin{pmatrix} \frac{q^{-2}}{[2]} & \frac{\sqrt{[3]}}{[2]} \\ \frac{\sqrt{[3]}}{[2]} & \frac{-q^2}{[2]} \end{pmatrix}.$$

## 4.4 Schur-Weyl duality

Let  $V$  be any finite-dimensional vector space over  $\mathbb{C}$ . Then  $V^{\otimes n}$  is a representation of  $\mathbb{C}[S(n)]$  via the simple permutation action

$$\pi(v_1 \dots v_n) = v_{\pi^{-1}(1)} \otimes \dots \otimes v_{\pi^{-1}(n)}. \quad (4.4.1)$$

The vector space  $V^{\otimes n}$  is also a representation of the Hecke algebra  $H_q(n)$  via a  $q$ -deformation of the permutation action defined in 4.4.1. In particular, the generator  $T_i$  acts on a vector in  $V^{\otimes n}$  by the identity on all factors except the  $i$ th and  $i + 1$ st ones. On these two factors, it acts by

$$T|v_i\rangle|v_j\rangle = \begin{cases} |v_j\rangle|v_i\rangle & \text{if } i < j, \\ (q^{-1} - q)|v_i\rangle|v_j\rangle + |v_j\rangle|v_i\rangle & \text{if } i > j, \\ q^{-1}|v_i\rangle|v_j\rangle & \text{if } i = j. \end{cases}$$

Note that when  $q = 1$ , we recover the the action defined by 4.4.1.

In the rest of this thesis we consider the case where  $V$  is the representation  $V^\lambda$  of  $U_q(d)$  indexed by the single-box partition  $\lambda = (1)$ . Using the coproduct struction on  $U_q(n)$ , we interpret  $V^{\otimes n}$  as a representation of  $U_q(n)$  as well as  $H_q(n)$ . In order for  $V^{\otimes n}$  to be a representation of the algebra  $U_q(d) \otimes H_q(n)$ , their respective actions must commute. This is proved by Jimbo in [7], as well as the following result which is known as *quantum Schur-Weyl duality*.

**Theorem 4.4.1.** *The space  $V^{\otimes n}$  as a representation of  $U_q(d) \otimes H_q(n)$  decomposes into irreps in the following formula*

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n, \ell(\lambda) \leq d} V^\lambda \otimes R^\lambda \quad (4.4.2)$$

Given that the basis of  $V$  is indexed by SSYT with a single box, we think of basis elements of  $V$  as letters, and thus the natural basis vectors of  $V^{\otimes n}$  as words of length  $n$ . This basis can be called the the *computational* or *word* basis for  $V^{\otimes n}$ .

With respect to the word basis for  $V^{\otimes n}$  and the GTJ and YYH bases for  $V^\lambda$  and  $R^\lambda$ , respectively, an algorithm carrying out the isomorphism given in 4.4.1 is

known as a *Schur-Weyl transform*. In the next chapter we describe a transform which decomposes  $V^{\otimes n}$  and prove that it is in fact a Schur-Weyl transform. The transform we define has a nice recursive structure, and we prove it has efficient time complexity in Chapter 6.

# Chapter 5

## The Pieri and Schur-Weyl transforms

### 5.1 Introduction

In Chapter 4 we presented the representation theories of the quantum group  $U_q(d)$  and the Hecke algebra  $H_q(n)$ , and their correspondence via Schur-Weyl duality. In this chapter we present a transform which we prove computes a Schur-Weyl transform. In Section 5.2 we define a Pieri transform via only the representation theory of the quantum group  $U_q(d)$ . In Section 5.3 we compose Pieri transforms and prove that this computes a Schur-Weyl transform. This result is stated without proof in [1], and we were unable to find a proof in the literature. Finally in Section 5.4 we look at the Pieri and Schur-Weyl transforms in their crystal limits, which ties representation theory together with the insertion algorithms seen in Section 3.3.

## 5.2 The Pieri transform

A *Pieri rule* is a formula for decomposing representations of  $U_q(d) \otimes U_q(d)$  that take the form  $V^\lambda \otimes V^{(m)}$ . In this chapter we will only need a Pieri rule for the case when  $m = 1$  so that we are decomposing  $V^\lambda \otimes V$ .

Although the inclusion  $U_q(d) \subseteq U_q(d) \otimes U_q(d)$  is not in general a Gelfand pair, in our case of interest the branching rule is multiplicity-free and is given in the following theorem.

**Theorem 5.2.1.** *Given the representation  $V^\lambda$  and the representation  $V = V^{(1)}$  of  $U_q(d)$ , their tensor product decomposes into irreps according to the following formula.*

$$V^\lambda \otimes V \cong \bigoplus_{\substack{\mu \text{ covers } \lambda \\ \ell(\lambda) \leq d}} V^\mu \quad (5.2.1)$$

An algorithm carrying out the isomorphism in equation 5.2.1 with respect to the GTJ bases for all  $U_q(d)$  representations is called a *Pieri transform*. Interpreting the Pieri transform at the level of tableaux, q-insertion becomes relevant. A basis element of  $V^\lambda \otimes V$  is indexed by an SSYT  $t$  and a letter  $i$ . The correct way of thinking of the Pieri transform is that it q-inserts  $i$  into  $t$  in superposition, resulting in elements which index vectors in  $V^\mu$  for  $\mu$  covering  $\lambda$ . A matrix entry of the Pieri transform is written  $\langle s \mid t, i \rangle$  and is non-zero only in the case when  $s$  is a result of q-inserting  $i$  into  $t$ . (This follows directly from the behavior of the generators of  $U_q(d)$ .) The Pieri coefficients are also called *Wigner coefficients*.

In the rest of this section we describe formulas for  $\langle s \mid t, i \rangle$ . The formulas can be found in a variety of sources, in particular in [3]. Similar to the GTJ coefficients described in Theorems 4.2.2 and 4.2.3, they are complicated-looking but in fact arise from simple combinatorial properties (e.g. axial distances) of the tableaux associated

to the basis elements.

As mentioned above, when computing  $\langle s \mid t, i \rangle$ , we visualize  $i$  as being  $q$ -inserted in  $t$ , activating a sequence of letter bumps. The first letter considered is  $i$ , and it must bump a larger letter until the last letter bumped is some letter  $i_k \leq d$ . WLOG we may assume that  $i_k = d$ , because if not we may work over the smaller algebra  $U_q(i_k)$ . Thus we can visualize a sequence of letters

$$i = i_1 < i_2 < \cdots < i_k = d \quad (5.2.2)$$

influenced by the bumping process. The Wigner coefficients factor into a product of *reduced Wigner coefficients*, one for each letter in 5.2.2. The letter  $i$  gets a special type of reduced Wigner coefficient, which we'll refer to as type zero.

**Theorem 5.2.2.** *Suppose  $i$  is  $q$ -inserted into the SSYT  $t$  into the box  $b$  in row  $r$ . The corresponding type zero reduced Wigner coefficient is given by*

$$W_0(i; \lambda^{(i)}) = q^{(res(b) - x_i(t) + 1)/2} \sqrt{\frac{1}{[a_{ir} + 1]} \prod_{j \leq i-1} \frac{[a_{jr} - \lambda_j^{(i)}]}{[a_{jr} + 1]}} \quad (5.2.3)$$

where  $a_{jk}$  are the axial distances with respect to the horizontal strip  $\lambda^{(i)}$ .

Any other letter  $k$  in the chain 5.2.2 is assigned a reduced Wigner coefficient of type one. Here we have two parameters: the box  $k$  inhabits in  $t$  in row  $r_1$  and the box  $k$  gets bumped into in tableau  $s$ , in row  $r_2$ . Then the type one Wigner coefficients are given in the following theorem.

**Theorem 5.2.3.** *Suppose  $i$  is  $q$ -inserted into the SSYT  $t$  and  $k$  gets bumped from row  $r_1$  to  $r_2$  as a result of the insertion. The corresponding type one reduced Wigner*

coefficient is given by

$$W_1(k; \lambda^{(k)}) = \text{sgn}(r_1 - r_2) q^{(a_{r_1 r_2} - \lambda_{r_1}^{(i)})/2} \sqrt{\prod_{\substack{j \leq k \\ j \neq r_2}} \frac{[a_{j r_1} + \lambda_{r_1}^{(k)} + 1]}{[a_{j r_2} + 1]} \prod_{\substack{j \leq k-1 \\ j \neq r_1}} \frac{[a_{j r_2} - \lambda_j^{(k)}]}{[a_{j r_1} + \lambda_{r_1}^{(k)} - \lambda_j^{(k)}]}} \quad (5.2.4)$$

where  $a_{jj'}$  are the axial distances with respect to the SSYT  $t^{(k)}$  and  $\text{sgn}(0) = 1$ .

Note that the exponent of  $q$  given by  $a_{r_1 r_2} - \lambda_{r_1}^{(i)}$  in theorem 5.2.3 is simply the distance from the old box  $k$  inhabited in  $t$  to the new box  $k$  inhabits in  $s$ .

**Theorem 5.2.4.** *The value of the Wigner coefficient  $\langle s \mid t, i \rangle$  with associated sequence (5.2.2) and with horizontal strips  $\lambda^{(i)} = \text{sh}(t^{(i)}) \setminus \text{sh}(t^{(i-1)})$  is given by*

$$\langle s \mid t, i \rangle = W_0(i) \prod_{j=2}^k W_1(i_j) \quad (5.2.5)$$

**Example 5.2.5.** Let  $t = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline \end{array}$  and  $s = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline \end{array}$  so that  $i = 2$  is the letter  $q$ -inserted into  $t$  to make  $s$ . There are two reduced Wigner coefficients, one associated to 2, and one associated to the 3 that 2 bumps.

In the first step, 2 is added onto the second row of the tableau  $\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline \end{array}$ . The residue of this box is zero,  $x_2(t) = 2$ , and the relevant axial distance is  $a_{12} = 3$ . Thus the type zero reduced Wigner coefficient associated to the 2 is given by

$$W_0(2) = q^{-1/2} \sqrt{[2]}$$

In the second step, the 3 is bumped from its original position in the second row to its new position in the third row. The axial distance from the old box to the new



box is given by 1. The relevant axial distances are  $a_{12} = 2$ ,  $a_{23} = 2$  and  $a_{13} = 4$ .

$$W_1(3) = -q^{1/2} \frac{[4]}{[3]} \sqrt{\frac{[2]}{[5]}}$$

Therefore, the Wigner coefficient  $\langle s \mid t, i \rangle$  is given by the product

$$W_0(2)W_1(3) = -\frac{[4][2]}{[3]\sqrt{[5]}}$$

Theorem 5.2.4 proves that the matrix entries of the Pieri transform decompose into a product of reduced Wigner coefficients. Another way of interpreting formula 5.2.5 is the following recursive version.

$$\langle s \mid t, i \rangle = \begin{cases} W_1(d) \langle s^{(d-1)} \mid t^{(d-1)}, i \rangle & i \neq d \\ W_0(d) & i = d \end{cases} \quad (5.2.6)$$

We then define the *reduced Wigner transform* to be an algorithm computing the  $d \times d$  matrix of reduced Wigner coefficients where  $t$  is fixed but  $i$  and the nonzero row in  $\text{sh}(s) \setminus \text{sh}(t)$  both vary.

### 5.3 The Schur-Weyl transform

Recall we use the notation  $V$  for the  $d$ -dimensional irrep of  $U_q(d)$  indexed by the single-box partition  $\lambda = (1)$ . Thus we think of basis elements of  $V$  as just letters and basis elements of  $V^{\otimes n}$  as words of length  $n$ .

For  $n = 2$ , we have  $V^{\otimes 2}$  which can be decomposed with a single Pieri transform seen in Section 5.2. For larger  $n$ , the Pieri transforms can be composed (or, cascaded) to create a transform with input space  $V^{\otimes n}$ . For example, when  $n = 3$ , we realize a

decomposition of  $V^{\otimes n}$  via two sequential applications of the Pieri transform.

$$\begin{aligned}
V^{\otimes 3} &= (V \otimes V) \otimes V \\
&\cong (V^{(1,1)} \oplus V^{(2)}) \otimes V \\
&\cong (V^{(1,1)} \otimes V) \oplus (V^{(2)} \otimes V) \\
&\cong (V^{(2,1)} \oplus V^{(1,1,1)}) \oplus (V^{(3)} \oplus V^{(2,1)}) \\
&\cong V^{(1,1,1)} \oplus (R^{(2,1)} \otimes V^{(2,1)}) \oplus V^{(3)}
\end{aligned}$$

The above description can be read as two sequential q-insertions mapping a word of length three to a tableau with three boxes. Note that there are two copies of  $V^{(2,1)}$  in the decomposition, determined by whether the box in the second row was added in the first or second instance of the Pieri transform, producing a multiplicity space  $R^{(2,1)}$  for the irrep  $V^{(2,1)}$ . As indicated by the notation, we know the multiplicity space  $R^{(2,1)}$  is isomorphic to the irrep  $R^{(2,1)}$  of the Hecke algebra by the Schur-Weyl duality theorem 4.4.1.

By induction, a decomposition of  $V^{\otimes n}$  can be achieved by cascading  $(n-1)$  Pieri transforms, which we refer to also as a Pieri transform. This transform results in a sum of  $U_q(d)$  irreps  $V^\lambda$ , whose multiplicity spaces are isomorphic to  $H_q(n)$  irreps  $R^\lambda$  via the Schur-Weyl duality theorem. However, a priori it is unclear whether the change-of-basis achieved by our Pieri transform is in fact identical to the change-of-basis required by a Schur-Weyl transform. In the rest of this section we prove that cascaded Pieri transforms compute the Schur-Weyl transform up to sign.

We refer to the GTT basis achieved by  $(n-1)$  Pieri transforms as the Pieri basis and the basis for Schur-Weyl duality as the Schur basis, as in Section 4.4.

**Theorem 5.3.1.** *The Schur transform for decomposing  $V^{\otimes n}$  by  $(n-1)$  cascaded Pieri*

*transforms is a Schur-Weyl transform as defined in Section 4.4 up to signs.*

*Proof.* We first argue the  $q = 1$  case, so that we can work with groups. Beginning with the group  $S(n) \times U(d)$ , in both the Pieri and Schur-Weyl decompositions we have subgroup flags that reach first  $U(d)$  to produce a summand of irreps  $V^\lambda$ . From this point we continue with the subgroup flag defined by the branching rule in theorem 4.2.1, to yield the standard GTT basis of  $V^\lambda$ .

In the Schur basis the subgroup flag is defined by

$$U(d) \subseteq S(2) \times U(d) \subseteq S(3) \times U(d) \subseteq \cdots \subseteq S(n) \times U(d)$$

and in the Pieri basis the subgroup flag is defined by

$$U(d) \subseteq U(d)^2 \subseteq U(d)^3 \subseteq \cdots \subseteq U(d)^n$$

In the Pieri flag, we mean more precisely that within the group  $U(d)^k$ , the first factor of  $U(d)$  should act on the first  $n - k + 1$  tensor factors of  $V$  diagonally, while the  $j$ th factor of  $U(d)$  for  $j \geq 2$  should act on the  $(n - k + j)$ th tensor factor of  $V$ . In other words, we can define the desired embedding  $U(d)^k \subseteq U(d)^{k+1}$  by the map

$$(\Delta \times \text{id}^{k-1}) : U(d)^k \rightarrow U(d)^{k+1},$$

where

$$\Delta : U(d) \rightarrow U(d)^2$$

is the standard diagonal embedding, and  $\text{id}$  is the identity.

We claim that the direct sum decompositions induced by the two subgroup flags become equal when they reach  $U(d)$  and irreps  $V^\lambda$  of this group. Since the subgroup

flags coincide below  $U(d)$ , the decomposition must remain equal afterwards. To prove the claim, we consider the partially ordered set  $P$  of groups  $S(j) \times U(d)^k$ , with  $j + k \leq n + 1$ , as shown in Figure 5.3.1.

$P$  has a unique minimal element,  $U(d)$ , and  $n$  maximal elements. It also has  $2^n$  maximal chains that connect  $U(d)$  to some maximal element, all of length  $n + 1$ ; each step of such a chain, in the figure, can either be down or to the right. We claim that each of these maximal chains are all locally multiplicity free on  $V^{\otimes n}$ , and that the partial decompositions all coincide when they reach  $U(d)$ .

To see that each chain  $c \subset P$  produces a GTT line basis, we first decompose  $V^{\otimes n}$  as a representation of a maximal group  $S(k) \times U(d)^{n-k+1}$ . By 4.4.1 we obtain the following isomorphism:

$$\begin{aligned} V^{\otimes n} &\cong V^{\otimes k} \otimes V^{\otimes n-k} \\ &\cong \left( \bigoplus_{\lambda \vdash k} R_\lambda \otimes V_\lambda \right) \otimes V^{\otimes n-k} \end{aligned}$$

This is multiplicity free. Each subsequent step of  $c$  is one of the inclusions

$$\begin{aligned} S(j-1) \times U(d)^k &\subseteq S(j) \times U(d)^k \\ S(j) \times U(d)^{k-1} &\subseteq S(j) \times U(d)^k \end{aligned}$$

Both of these inclusions are Gelfand pairs by Theorems 4.2.1 and 4.3.1, and the structure of irreps of the direct product of two groups.

To see that the decompositions coincide, we consider two types of moves on chains in  $P$ : a triangle move that changes the last step between horizontal and vertical, and a square move that switches a horizontal and vertical step lower in the chain. The

triangle move relates two chains  $c_1, c_2 \subseteq P$  that agree except at the three groups:

$$\begin{array}{ccc} S(k) \times U(d)^{n-k} & \subseteq & S(k+1) \times U(d)^{n-k} \\ \text{I} \cap & & \\ S(k) \times U(d)^{n-k+1} & & \end{array}$$

We claim that the decomposition of  $V^{\otimes n-k}$  is multiplicity free using the chain  $c_3 = c_1 \cap c_2$ , which begins directly with  $S(k) \times U(d)^{n-k}$ . By 4.4.1 and either Theorem 4.2.1 or Theorem 4.3.1, we obtain

$$V^{\otimes n} \cong \left( \bigoplus_{\substack{\lambda \vdash k, \mu \vdash k+1 \\ \mu \text{ covers } \lambda}} R^\lambda \otimes V^\mu \right) \otimes V^{\otimes n-k-1},$$

which is multiplicity free. Since  $c_1$  and  $c_2$  each yield the same decomposition as  $c_3$ , they yield the same decomposition as each other.

Likewise suppose that  $c_1, c_2 \subseteq P$  differ by a square move:

$$\begin{array}{ccc} S(j) \times U(d)^k & \subseteq & S(j+1) \times U(d)^k \\ \text{I} \cap & & \text{I} \cap \\ S(j+1) \times U(d)^k & \subseteq & S(j+1) \times U(d)^{k+1} \end{array}.$$

We claim that  $c_3 = c_1 \cap c_2$  is again locally multiplicity free, which implies that  $c_1$  and  $c_2$  must each yield the same decomposition as  $c_3$ . At the lower left corner, a single summand which is an irrep of  $S(j+1) \times U(d)^{k+1}$  will in general have the form  $(R_\lambda \otimes V_\mu) \otimes V^{\otimes k}$ . Then its restriction to  $S(j) \times U(d)^k$  is multiplicity free, by applying Theorem 4.3.1 to  $R_\lambda$  and Theorem 4.2.1 to  $V_\mu \otimes V$ .

It is easy to see that all maximal chains in  $P$  are connected by square and triangle moves. This yields the result when  $q = 1$ .

$$\begin{array}{ccccccc}
U(d) & \subseteq & S(2) \times U(d) & \subseteq & S(3) \times U(d) & \subseteq & \cdots \subseteq S(n-2) \times U(d) \subseteq S(n-1) \times U(d) \subseteq S(n) \times U(d) \\
\cap & & \cap & & \cap & & \cap \\
U(d)^2 & \subseteq & S(2) \times U(d)^2 & \subseteq & S(3) \times U(d)^2 & \subseteq & \cdots \subseteq S(n-2) \times U(d)^2 \subseteq S(n-1) \times U(d)^2 \\
\cap & & \cap & & \cap & & \cap \\
U(d)^3 & \subseteq & S(2) \times U(d)^3 & \subseteq & S(3) \times U(d)^3 & \subseteq & \cdots \subseteq S(n-2) \times U(d)^3 \\
\cap & & \cap & & \cap & & \\
\vdots & & \vdots & & \vdots & & \ddots \\
\cap & & \cap & & \cap & & \\
U(d)^{n-2} & \subseteq & S(2) \times U(d)^{n-2} & \subseteq & S(3) \times U(d)^{n-2} & & \\
\cap & & \cap & & & & \\
U(d)^{n-1} & \subseteq & S(2) \times U(d)^{n-1} & & & & \\
\cap & & & & & & \\
U(d)^n & & & & & & 
\end{array}$$

Figure 5.3.1: The poset  $P$  of group inclusions

When  $q$  is positive (or more generally, when  $q$  is not a root of unity) we can follow the same argument, except that we replace  $S(j) \times U(d)^k$  by  $H_q(j) \otimes U_q(d)^{\otimes k}$ . The replacement yields well-defined algebra actions by 4.4.1, and the argument still works because it relies on the same multiplicity free structures.

The line basis agreement extends to unique vector bases up to sign. Because the representations considered are all bar representations, the same Gelfand-Tsetlin constructions yield unique real line bases. Then, because the representations are all \*-representations, the real line bases can be refined to bases of real unit vectors. These vectors are then unique up to sign.

□

## 5.4 Pieri and Schur-Weyl in the crystal limit

In quantum algebra the limit  $q = 0$  is referred to as the *crystal limit*. The quantum groups and Hecke algebras we have considered do not have well-defined algebra structures at  $q = 0$ . However, it can be useful to look at the transforms in the crystal limit, as they are still linear maps between vector spaces, if not representation

isomorphisms. In this section we describe the behavior of the Wigner coefficients defined in the Pieri transform in Section 5.2 in the crystal limit. These theorems can be found in paper [3].

**Theorem 5.4.1.** *The type zero Wigner coefficient  $W_0(r; \lambda^{(i)})$  is zero in the limit  $q = 0$  except when for every  $j$  between  $r$  and  $i - 1$  we have  $\lambda_{j+1}^{(i)} = \lambda_j^{(i-1)}$ . The type zero Wigner coefficient  $W_0(r; \lambda^{(i)})$  is zero in the limit  $q = \infty$  except when  $r = 1$ , and in this case it's one.*

**Theorem 5.4.2.** *The type one Wigner coefficient  $W_1(r_1; r_2; \lambda^{(i)})$  is zero in the limit  $q = 0$  for every choice except when  $r_1 = r_2$  or when  $r_1 > r_2$  and  $\lambda_{j+1}^{(i)} = \lambda_j^{(i-1)}$  for  $j$  between  $r_2$  and  $r_1 - 1$ , and in both these cases it's 1. The type one Wigner coefficient  $W_1(r_1; r_2; \lambda^{(i)})$  is zero in the limit  $q = \infty$  for every choice except when  $r_2 = r_1 + 1$ , and in this case it's  $-1$ .*

At the level of tableaux, theorems 5.4.1 and 5.4.2 can be interpreted in the following way.

In the limit  $q = 0$ , the only time a letter can be inserted into a tableau without bumping another letter is when it is added to the very first column that does not contain an  $i$ . Also, the only time a letter can be bumped, it gets inserted into the very next column. This is a description of the dual RSK insertion algorithm.

In the limit  $q = \infty$  the only time a letter can be inserted into a tableau without bumping another letter is when it can be added to the first row, and the only time a letter can be bumped, it gets inserted into the following row. This is a description of the RSK insertion algorithm together with a bumping sign described in section 3.3.

Therefore, in the crystal limits  $q = 0$  and  $q = \infty$  the transform defined in 5.2.1 is the dual RSK and RSK insertion algorithm with bumping sign, respectively. In

this sense, RSK and dual RSK are ‘classical’ versions of quantum insertion (not to be confused with the other notion of ‘classical’ in this setting, i.e.  $q = 1$ ).

Extending Theorems [5.4.1](#) and [5.4.2](#) to cascaded Pieri transforms, we conclude that in the crystal limits, the Schur-Weyl transform computes the well-known RSK bijections on finite sets described in Section [3.3](#).



# Chapter 6

## A quantum algorithm for the quantum Schur transform

### 6.1 Introduction

In this chapter we present our main theorem, which is a quantum algorithm for computing Schur-Weyl duality. A quantum computer uses basic units of information called *qubits* rather than bits. Since the algebra of qubit spaces is quite different than that of bits, the possible types of qubit transformations, and thus the maps considered by quantum computers in their calculations, are different as well. This in turn influences the types of things that can be calculated within a certain time complexity.

In Section 6.2 we describe the basics of quantum probability, which forms the algebraic base for quantum computation. In Section 6.3 we review the basic theory of quantum algorithms, and in Section 6.4 we present our main theorem. The methods we use in the proof of our main theorem are modeled on those found in the paper

[1], where they prove the existence of a Schur-Weyl algorithm for the case  $q = 1$ . As far as we know, this thesis contains the first instance of an algorithm designed for a quantum computer for the purpose of decomposing quantum algebra representations.

## 6.2 Quantum probability

In this section we describe some elements of quantum probability and their relation to quantum algorithms. A thorough treatment of this material can be found in the book [13].

We fix the *computational basis* of  $\mathbb{C}^d$  to consist of the orthonormal vectors  $|1\rangle, \dots, |d\rangle$ , and write all matrices in this basis. The algebra  $M_d$  is the set of all  $d \times d$  matrices in this basis. (The more standard numbering in computer science is from 0 to  $d-1$ ; we use the mathematicians' numbering which is more standard in combinatorial representation theory.)

When  $d = 2$ ,  $M_2$  is called a *qubit*, and represents the probability space corresponding to a quantum particle with two basis states, such as an electron which can be measured as either spin-up or spin-down. For larger  $d$ ,  $M_d$  is called a *qudit*. Joint systems are constructed by tensoring qudits. So, for example, a pair of qubits is  $M_2 \otimes M_2 \cong M_4$ .

The *state* of a qudit  $M_d$  is defined to be a positive and normalized dual vector on  $M_d$ . There is an isomorphism between  $M_d$  and its dual space so we can view dual vectors of  $M_d$  as elements of  $M_d$  itself. Under this isomorphism,  $\rho \in M_d$  acts as a dual vector on  $M_d$  according to the formula

$$\rho(a) = \text{Tr}(\rho a) \tag{6.2.1}$$

The *pure* states of  $M_d$  are of the form  $\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|$ , where  $|\psi\rangle$  is a normalized vector in  $\mathbb{C}^d$ . Following Equation 6.2.1, its action on an element  $b \in M_d$  is given by the inner product

$$\rho_{|\psi\rangle}(b) = \text{Tr}(|\psi\rangle\langle\psi|b) = \langle\psi|b|\psi\rangle.$$

Thus pure states are indexed by normalized vectors in  $\mathbb{C}^d$ .

The self-adjoint elements in  $M_d$ , i.e. those that satisfy  $a^* = a$ , are called *observables* or *measurables*. The *idempotent* observables, sometimes called *events*, satisfy  $a^2 = a$ , and are interpreted as measuring whether the qudit is in the state  $a$ . By the Spectral theorem, any observable  $a$  can be decomposed into a sum of idempotent observables:

$$a = \sum_{\lambda \in \sigma(a)} \lambda a_\lambda, \quad (6.2.2)$$

where  $\sigma(a)$  is the spectrum of  $a$ , and  $a_\lambda$  is the projection operator onto the eigenspace defined by  $\lambda$ . The Spectral theorem guarantees a choice of projection operators  $a_\lambda$  which are pairwise orthogonal. The probability that the observable  $a$  measures  $\lambda$  is

$$\text{Prob}[a = \lambda] = \text{Tr}(\rho a_\lambda) \quad (6.2.3)$$

and the state of the qudit passes to the conditional state  $\rho_\lambda = \rho_{|\phi\rangle}$ , which is defined by

$$|\phi\rangle = \frac{a_\lambda |\psi\rangle}{\sqrt{\langle\psi|a_\lambda|\psi\rangle}} \quad (6.2.4)$$

We will mainly use the special case  $a = \sum_k |k\rangle\langle k|$ , which is a *complete* measure-

ment in the computational basis. Expanding the vector  $|\psi\rangle = \sum_k \psi_k |k\rangle$ , we have

$$\rho_{|\psi\rangle} = \sum_{j,k} \rho_{j,k} |j\rangle\langle k|,$$

where  $\rho_{j,k} = \psi_j^* \psi_k$ . Therefore, following Equation 6.2.3, the probability that the state of the qudit is measured to be  $|k\rangle$  is  $\rho_{k,k} = |\psi_k|^2$ , and following Equation 6.2.4, the state then passes to  $\rho_k = |k\rangle\langle k|$ . So, after a complete measurement the state of a qudit is defined by one of the basis vectors  $|k\rangle$ , which can be used as an ‘answer’ to a computational question.

The state of a qudit can also undergo reversible unitary *evolution*  $E$  given by conjugations  $E(\rho) = U\rho U^{-1}$  where  $U \in PSU(d)$ , the space of projective unitary maps. If  $\rho_{|\psi\rangle}$  is a pure state, then

$$E(\rho_{|\psi\rangle}) = U\rho_{|\psi\rangle}U^{-1} = \rho_{U|\psi\rangle},$$

is also pure and defined by the action of an element in  $PSU(d)$ .

Pure states, measurement operators, and unitary evolution are the basic elements necessary to define quantum algorithms, which we describe in Section 6.3.

## 6.3 Quantum algorithms

In this section we review basic quantum algorithms. For a more complete introduction, refer again to [13].

In the context of quantum computing, unitary operators acting on qudit spaces are called *quantum gates*. In the usual interpretation, a quantum gate acting on  $m$  qudits can act on  $n \geq m$  qudits by acting by  $U$  on  $m$  of the qudits and the identity

on the other  $n - m$  qudits. A *quantum circuit* is a composition of quantum gates. The *time complexity* of a quantum circuit is the number of quantum gates in its decomposition. We require that a family of quantum circuits be *uniform*, meaning there is a classical algorithm to compute the decomposition of the quantum circuits into quantum gates. The time complexity of this decomposition algorithm is counted toward the total time complexity of the quantum algorithm.

Because the set of quantum gates is infinite, but a quantum computer would have access to a finite number of quantum gates, it's generally not possible to construct an exact quantum circuit for calculating a given unitary transformation. In other words, some approximation will usually be necessary.

A finite set of gates  $G$  that acts on at most  $m$  qudits of size  $d$  is called *universal* if it generates a dense subgroup of  $U(d^m)$  for some  $m \geq 2$ ; it consequently densely generates  $U(d^n)$  for any  $n \geq m$ . In other words, a set  $G$  of quantum gates is universal if every unitary operator  $A \in U(d^n)$  can be approximated by a quantum circuit composed of elements from  $G$ .

The Solovay-Kitaev theorem [4] establishes that any operator  $A \in U(d)$  can be approximated by words in a universal gate set with time and gate complexity  $\text{poly}(d, \log \epsilon^{-1})$ , where  $\epsilon$  is the error of the approximation. This is provided that the matrix entries of the gates and the matrix entries of  $A$  can be approximated with the same time complexity. Thus, up to polylogarithmic overhead, any universal gate set is equivalent to all unitary operators in  $U(d)$  whose matrix entries can be computed quickly. Often this theorem is used for fixed values of  $d$ , but the algorithm is constructive and it is easy to establish that the gate complexity (and classical time complexity to choose the gates) is polynomial in  $d$  as well. The Solovay-Kitaev theorem establishes that this time complexity is independent of the universal gate set up

to a polylogarithmic factor (assuming that the matrices of the gates can be computed quickly).

Both classically and quantumly, an *efficient* algorithm is one which has polynomial time complexity. The class of decision problems with efficient classical probabilistic algorithms is called *BPP*, for bounded-error polynomial time. The class of decision problems with efficient quantum algorithms is analogously called *BQP*, for bounded-error quantum polynomial time. The class BQP contains the class BPP, meaning that quantum algorithms can efficiently solve any decision problems that classical algorithms can solve. Whether or not BQP is strictly larger than BPP remains an open problem. However, there are decision problems provably in BQP that are not provably in BPP, such as the factoring problem investigated in [16].

## 6.4 Time complexity of Schur-Weyl

In this section we describe the time complexity of computing first a single Pieri transform, and then the cascade of Pieri transforms which calculates a Schur-Weyl transform.

In Equation 5.2.5, we calculated the matrix entry  $\langle s \mid t, i \rangle$  of the Pieri transform in terms of reduced Wigner coefficients. Using Formulas 5.2.2 and 5.2.3 with 5.2.5 allow us to calculate all the matrix entries of the Pieri transform, but at this point a direct application of the Solovay-Kitaev theorem is not quite sufficient, since the entire Pieri operator has a matrix of size  $d|\text{SSYT}(\lambda)|$ , which is larger than  $\text{poly}(n, d)$ .

This is why we use the recursive version of 5.2.5 given by 5.2.6 to define the reduced Wigner transform as the  $d \times d$  matrix of reduced Wigner coefficients where  $t$  is fixed. As described in [1], given an input  $t$ , the coefficients in the reduced Wigner transform are calculated by a classical algorithm and then the reduced Wigner transform is

calculated by a quantum algorithm.

**Theorem 6.4.1.** *There is a quantum algorithm for computing the controlled reduced Wigner transform with time complexity  $\text{poly}(n, d, \log \epsilon^{-1})$ , where  $\epsilon$  is the desired accuracy.*

Combining the recursive formula 5.2.6 with Theorem 6.4.1, we obtain the following result.

**Theorem 6.4.2.** *There is a quantum algorithm to compute the quantum Pieri transform with accuracy  $\epsilon$  with time complexity  $\text{poly}(n, d, \log \epsilon^{-1})$ , where  $\epsilon$  is the desired accuracy.*

As proved in section 5.3, a Schur-Weyl transform can be computed up to sign using  $(n - 1)$  cascaded Pieri transforms. We use this fact combined with theorem 6.4.2 to calculate the total time complexity of the Schur-Weyl transform given in the following theorem.

**Theorem 6.4.3.** *There is a quantum algorithm for computing the cascaded Pieri transform which is a Schur-Weyl transform on  $V^{\otimes n}$  with time complexity  $\text{poly}(n, d, \log(\epsilon^{-1}))$ , where  $\epsilon$  is the desired accuracy.*

Since Theorems 6.4.2 and 6.4.3 are entirely based on unitary operators for any real number  $q > 0$ , and since unitary groups are compact, we can expect the Schur algorithm to have well-defined limits at  $q = 0$  and  $q = \infty$ , provided that we keep  $q$  real and positive. This expectation turns out to be correct. As proved in [3] and discussed in Section 5.4, in the limit  $q = 0$  the transforms converge to permutation matrices, while in the limit  $q = \infty$ , they converge to signed permutation matrices.

Therefore, at  $q = 0$  and  $q = \infty$  our algorithm is a unitary version of the dual RSK and RSK algorithm combined with a bumping sign algorithm.

# Chapter 7

## Conclusion

We conclude this thesis by discussing some directions for further research into the topic of Schur transforms.

First of all, it is still open whether there is an efficient algorithm for the Schur-Weyl transform for any value of  $q$ , including  $q = 1$ , which is jointly polynomial in  $n$  and  $\log d$ , in other words polynomial in the input qubit length  $n(\log d)$ .

A second line of investigation involves finding algorithms for other transforms of quantum algebras. In particular, the algorithm in [2] which is an efficient computation for the quantum Fourier transform for the symmetric group  $S(n)$  could possibly be generalized to a quantum Fourier-like transform decomposing the regular representation of the Hecke algebra  $H_q(n)$ .

The quantum Fourier transform decomposes a representation of  $S(n) \times S(n)$ , while the Schur transform decomposes a representation of  $S(n) \times U(d)$ . There is a known algebra isomorphism for decomposing an analogous representation of  $U(d) \times U(d)$  known as Howe duality, which is extended to the quantum algebra  $U_q(d)$  in [18]. Whether there is an efficient quantum transform for computing a version of Howe duality for any value of  $q$ , including  $q = 1$ , is open. A possible goal is to generalize



all three algorithms (Fourier, Schur, and Howe) into a single algorithm.

Finally, one may investigate the types of applications  $q$ -deformed quantum algorithms can solve. One of the main reasons that quantum Fourier transforms have been studied so extensively is their link to hidden subgroup problems which are in turn linked to interesting computational problems such as factoring and graph isomorphism. In [1] some applications of the Schur transform for  $q = 1$  are proposed. The types of computational problems quantum algorithms for decomposing quantum algebra representations help to solve remain to be investigated.

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